Projections

A First Look

Brian D. Fitzpatrick

October 4, 2026

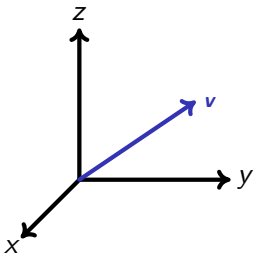
Duke UNIVERSITY MATH

Geometric Motivation

Projecting to z-Axis

Question

What is the *projection* $P\boldsymbol{v}$ of $\boldsymbol{v}$ onto the z-axis?



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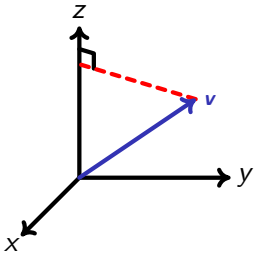

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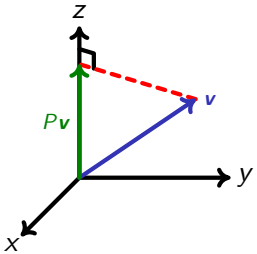

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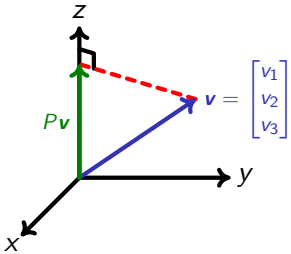

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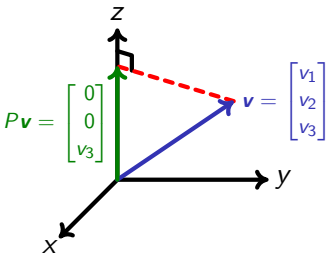

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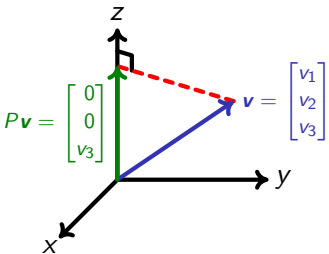

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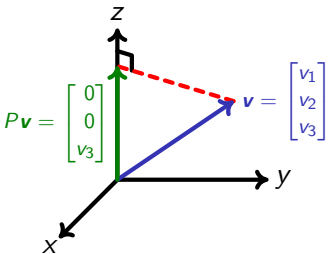

$$\overset{P}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\boldsymbol{v}}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}$$

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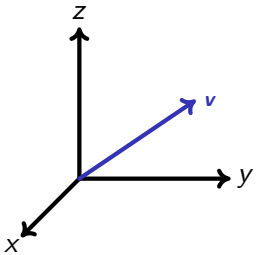

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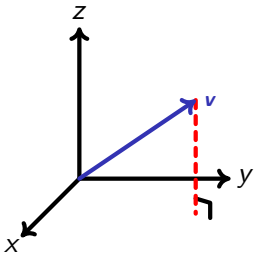

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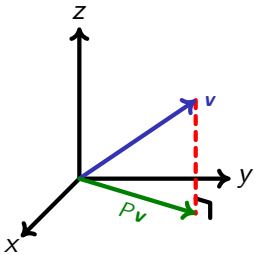

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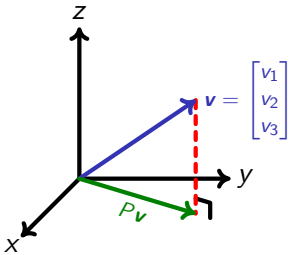

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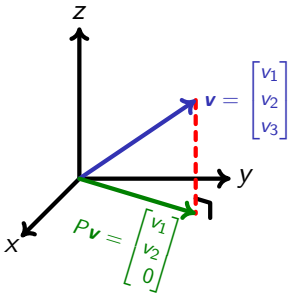

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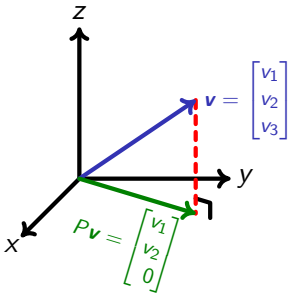

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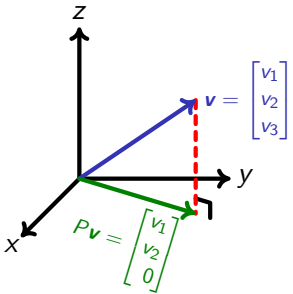

$$\overset{P}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \overset{\boldsymbol{v}}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$$

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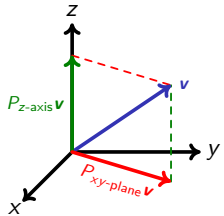
Geometric Motivation

Adding the Projections

Observation

Note that $P_{z\text{-axis}} + P_{xy\text{-plane}} = I_3$.



$$\overset{P_{z\text{-axis}}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} + \overset{P_{xy\text{-plane}}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}} = \overset{I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

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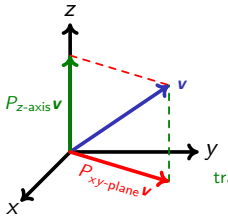

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_{z\text{-axis}}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{P_{xy\text{-plane}}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3}$$

$\text{trace}(P_{z\text{-axis}}) = \dim(z\text{-axis})$

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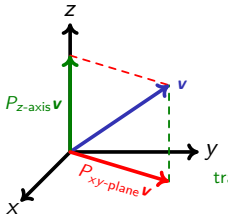

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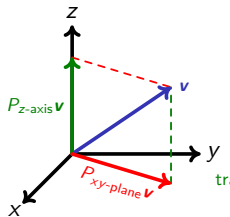

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$\text{trace}(P_{z\text{-axis}}) = \dim(z\text{-axis})$

$\text{trace}(P_{xy\text{-plane}}) = \dim(xy\text{-plane})$

Also note that the z-axis and xy-plane are *orthogonal complements*.

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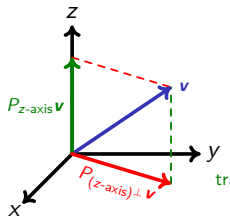
Note that $P_{z\text{-axis}} + P_{(z\text{-axis})^\perp} = I_3$.



$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_{z\text{-axis}}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{P_{(z\text{-axis})^\perp}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3}$$

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